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SQUEEZED STATES IN THE THEORY OF PRIMORDIAL GRAVITATIONAL WAVES

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ABSTRACT

It is shown that squeezed states of primordial gravitational waves are inevitably produced in the course of cosmological evolution. The theory of squeezed gravitons is very similar to the theory of squeezed light. Squeeze parameters and statistical properties of the expected relic gravity-wave radiation are described.

INTRODUCTION

Squeezed quantum states of light have been successfully generated and detected under laboratory conditions. It is known how much skill and effort by our experimentalist colleagues it requires to achieve even a modest amount of squeezing, that is, to obtain the squeeze parameter r of order of 1. The main purpose of my talk is to show that, in the cosmos, squeezed quantum states of gravitational waves are produced inevitably and with a much much larger amount of squeezing, simply as a result of expansion of the Universe.

In the context of gravity-wave research, the notion of squeezed quantum states has been often referred to. However, what was always meant was the squeezing of a quantized vibrational mode of a detecting device that could be implemented for a better detection of a classical gravitational wave. For instance, it was shown (Ref. 1) that the performance of a laser interferometer gravity-wave detector can be improved by using squeezed light. In another paper (Ref. 2) it was argued that any detector-oscillator can be specially "prepared" in a squeezed state and used for gravity-wave detection during some interval of time before the thermal noise destroys squeezing and degrades the detector's sensitivity.

However, it is the squeezing of the gravitational waves themselves that will be discussed in my talk today. I will show that the production of squeezed relic gravitational waves is an inescapable consequence of the variability of cosmological gravitational field and the existence of zero-point quantum fluctuations.

The mathematical theories of relic graviton production and squeezing of light are very similar. To make this similarity especially transparent, I will begin by presenting Einstein's general relativity in the form of a traditional field theory, such as the theory of classical electromagnetic fields. Those of you who may feel uncomfortable, or even intimidated, with the notion of curved space-time, will, perhaps, find it easier to deal with the concept of a gravitational field given in the usual flat Minkowski space-time. (More details about this "field-theoretical" formulation of general relativity are presented in Ref. 3; it is important to emphasize that we are dealing with a different mathematical formulation of general-

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relativity, not with some alternate physical theory, see Ref. 4.) This approach leads to manifestly nonlinear field equations. In contrast to quantum optics based on the laws of linear electrodynamics, in the case of a gravitational field one does not need any material medium in order to couple a "pump" field with the "signal" waves; for gravity this is achieved automatically due to the nonlinearity of the gravitational field itself. As is often done, we will present the total gravitational field in the form of an approximate sum of a large "classical" contribution and a small quantized perturbation. This approach will be applied to the cosmological gravitational field of the expanding Universe acting upon zero-point quantum fluctuations of the gravitational waves. I hate to call the variable gravitational field of the expanding Universe—the most grandiose and magnificent phenomenon we are aware of—just a "pump" field, but, technically speaking, it plays precisely this role. As a result, the initial vacuum state of gravitational waves will evolve into a strongly squeezed vacuum state with very specific statistical properties.

We will discuss the expected characteristics of the relic gravity-wave background radiation and the problem of its detection.

FIELD-THEORETICAL APPROACH TO GENERAL RELATIVITY

A gravitational field is fully described by a symmetric second-rank tensor $h_{\mu\nu}$ (note that the gravitational field variables have just one extra index as compared with the electromagnetic 4-vector potential A_{μ} , not a big difference!). For writing down the Lagrangian of the gravitational field it is convenient to use also an additional set of variables: the tensor field $P^{\alpha}_{\ \mu\nu}$, symmetric with respect to the last two indices. However, $P^{\alpha}_{\ \mu\nu}$ is not a new physical field but rather a combination of the first derivatives of $h_{\mu\nu}$, as follows from the field equations.

Gravitational field potentials $h_{\mu\nu}(x,y,z,t)$ are mathematically treated as components of one of many physical fields immersed in the ordinary Minkowski space-time:

$$d\sigma^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$
(1)

The metric tensor of Minkowski space-time will be denoted by $\gamma_{\mu\nu}$. With respect to this metric tensor all covariant differentiations (denoted by the symbol ";") and lowerings or raisings of indices are to be performed. In the Lorentzian coordinates, like the ones implied in eq. (1) and which we will be using in practical calculations below, $\gamma_{\mu\nu}$ acquires especially simple values: $\gamma_{00} = 1$, $\gamma_{11} = \gamma_{22} = \gamma_{33} = -1$, with the rest of $\gamma_{\mu\nu}$ being equal to zero.

The gravitational part of the total action $S = S^g + S^m$ is

$$S^g = -\frac{1}{2c\kappa} \int d^4x L^g , \quad \kappa = \frac{8\pi G}{c^4}$$

where the gravitational Lagrangian L^g has the form

$$L^g = (-\gamma)^{1/2} \left[h^{\mu\nu}_{;\alpha} P^{\alpha}_{\mu\nu} - (h^{\mu\nu} + \gamma^{\mu\nu}) \left(P^{\alpha}_{\mu\beta} P^{\beta}_{\nu\alpha} - \frac{1}{3} P^{\sigma}_{\sigma\mu} P^{\rho}_{\rho\nu} \right) \right] \ . \label{eq:Lg}$$

The nongravitational sources and fields and their interaction with the gravitational field are described by

$$S^m = \frac{1}{c} \int d^4x \ L^m \ .$$

The energy-momentum tensor $t_{\mu\nu}$ of the gravitational field itself and the energy-momentum tensor $\tau_{\mu\nu}$ of the nongravitational matter and fields interacting with gravity are defined in the usual manner:

$$\kappa t_{\mu\nu} = -\frac{1}{\sqrt{-\gamma}} \; \frac{\delta L^g}{\delta \gamma^{\mu\nu}} \quad , \quad \tau_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \; \frac{\delta L^m}{\delta \gamma^{\mu\nu}} \; . \label{eq:kappa}$$

The precise expression for $\kappa t_{\mu\nu}$ is as follows:

$$\kappa t_{\mu\nu} = P^{\alpha}_{\ \mu\beta} P^{\beta}_{\ \nu\alpha} - \frac{1}{2} \gamma_{\mu\nu} \gamma^{\sigma\rho} P^{\alpha}_{\ \sigma\beta} P^{\beta}_{\ \rho\alpha} - \frac{1}{3} \left(P_{\mu} P_{\nu} - \frac{1}{2} \gamma_{\mu\nu} P_{\sigma} P^{\sigma} \right) + Q_{\mu\nu} \tag{2}$$

where $P_{\mu} \equiv P^{\alpha}_{\alpha\mu},\, Q_{\mu\nu} \equiv Q^{\alpha}_{\mu\nu;\alpha}$ and

$$Q^{\tau}_{\ \mu\nu} \equiv \frac{1}{2} P^{\sigma}_{\ \alpha\beta} \left[\gamma_{\mu\nu} h^{\alpha\beta} \delta^{\tau}_{\sigma} + \gamma^{\tau\alpha} \left(\gamma_{\sigma\mu} h^{\beta}_{\nu} + \gamma_{\sigma\nu} h^{\beta}_{\mu} \right) - h^{\tau\beta} \left(\delta^{\alpha}_{\mu} \gamma_{\sigma\nu} + \delta^{\alpha}_{\nu} \gamma_{\sigma\mu} \right) - \delta^{\tau}_{\sigma} \left(\delta^{\alpha}_{\mu} h^{\beta}_{\nu} + \delta^{\alpha}_{\nu} h^{\beta}_{\mu} \right) \right] \ .$$

By varying the action S with respect to $h^{\mu\nu}$ and $P^{\alpha}_{\ \mu\nu}$ one can derive the following gravitational field equations:

$$h_{\mu\nu}^{\ ,\alpha}_{\ ,\alpha} + \gamma_{\mu\nu} h^{\alpha\beta}_{\ ,\alpha,\beta} - h^{\alpha}_{\ \nu,\alpha,\mu} - h^{\alpha}_{\ \mu,\alpha,\nu} = \frac{16\pi G}{c^4} (t_{\mu\nu} + \tau_{\mu\nu})$$
 (3)

where the comma means an ordinary derivative $(\alpha = \partial/\partial x^{\alpha})$ and $\gamma_{\mu\nu}$ is assumed to be in the simplest form corresponding to eq. (1). For reference purposes we will also write down the relationship between the first derivatives of $h_{\mu\nu}$ and $P^{\alpha}_{\mu\nu}$:

$$-h^{\mu\nu}_{;\rho} - \frac{1}{2} h^{\mu\nu} \gamma^{\alpha\beta} \ \gamma_{\alpha\beta,\rho} + \left(\gamma^{\mu\alpha} + \ h^{\mu\alpha}\right) P^{\nu}_{\alpha\rho} + \left(\gamma^{\nu\alpha} + \ h^{\nu\alpha}\right) P^{\mu}_{\alpha\rho} - \frac{1}{3} P_{\alpha} \left[\left(\gamma^{\mu\alpha} + \ h^{\mu\alpha}\right) \delta^{\nu}_{\rho} + \left(\gamma^{\nu\alpha} + \ h^{\nu\alpha}\right) \delta^{\mu}_{\rho} \right] = 0.$$

The theory possesses a gauge freedom quite similar to the gauge freedom of classical electrodynamics. One can apply the gauge transformations to the gravitational variables $h_{\mu\nu}$ and matter variables without changing the field equations. At the expense of the gauge freedom one can impose some gauge conditions which are normally used for diminishing the number of variables and simplifying the field equations.

The transition to the usual "geometrical" formulation of general relativity is established by introducing the new functions $g_{\mu\nu}$ according to the rule

$$\sqrt{-g} g^{\mu\nu} = \sqrt{-\gamma} \left(\gamma^{\mu\nu} + h^{\mu\nu} \right) \tag{4}$$

and by identifying the $g_{\mu\nu}$ with the metric tensor of the curved space-time: $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$.

GRAVITATIONAL FIELD OF THE EXPANDING UNIVERSE

Let us apply the developed formalism for description of the gravitational field of the homogeneous isotropic Universe. From our new point of view this is just a specific gravitational field $h_{\mu\nu}(t,x,y,z)$ given in Minkowski space-time (1). Let us take the nonvanishing gravitational potentials in the form

$$h_{00} = a^3(t) - 1$$
 , $h_{11} = h_{22} = h_{33} = 1 - a(t)$ (5)

where a(t) is, as yet unspecified, function of time. One can calculate the gravitational energy-momentum tensor $t_{\mu\nu}$, eq. (2), and find that the nonvanishing components of $\kappa t_{\mu\nu}$ are

$$\kappa t_{00} = -\frac{3\ddot{a}}{2a}(a^2 - 1) - 3\dot{a}^2$$

$$\kappa t_{11} = \kappa t_{22} = \kappa t_{33} = -\frac{\ddot{a}}{2a}(3a^3 - a^2 + a - 3) - \dot{a}^2(3a - 1)$$
(6)

where the dot means the time derivative (for simplicity we choose the units where the velocity of light c=1).

The nongravitational sources are assumed to be in the form of hydrodynamical matter with the Lagrangian

$$L^{m} = \frac{1}{2}\sqrt{-g} \left[\epsilon + 3p - (\epsilon + p)g_{\mu\nu} u^{\mu} u^{\nu}\right]$$

where $g_{\mu\nu}$ is defined by eq. (4). One can find the nonvanishing components of the energy-momentum tensor $\tau_{\mu\nu}$:

$$\tau_{00} = \epsilon + \frac{3}{4}(a^2 - 1)(\epsilon - p)$$

$$\tau_{11} = \tau_{22} = \tau_{33} = p - \frac{1}{4}(a^2 - 1)(\epsilon - p).$$
(7)

By substituting expressions (5), (6), and (7) into the field equations (3) one can derive equations governing the function a(t) and, hence, the gravitational field (5):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\epsilon + 3p), \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\epsilon.$$

(In "geometrical" language, these are, of course, the Einstein equations for a spatially flat cosmological model: $ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2)$.) By specifying the relationship between ϵ and p ("the equation of state") one can solve these equations and find a concrete function a(t).

AMPLIFICATION OF GRAVITATIONAL WAVES

Gravitational field (5) is just a main term of a more complicated and realistic cosmological gravitational field which includes the gravity-wave perturbations. Let us present the total field $h_{\mu\nu}$ in the form

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(1)}(t, x, y, z) , \qquad (8)$$

where $h_{\mu\nu}^{(0)}$ is given by eq. (5). At the expense of gauge freedom one can always satisfy the conditions $h_{\nu\nu}^{(1)\mu\nu}=0$. Moreover, in the case of gravitational waves, the perturbations of ϵ , p, and u^{α} are equal to zero and one can, in addition, satisfy the requirements $h_{o\mu}^{(1)}=0$, $h_{\mu\nu}^{(1)}\gamma^{\mu\nu}=0$, so that one is left with only two independent polarization components (designated by a=1,2) of $h_{\mu\nu}^{(1)}$. For a wave with the wave vector \mathbf{n} one can write down the nonzero components of the field:

$$h_{ij}^{(1)}(t,x,y,z) = \left(\mu_n^a(t)e^{i\mathbf{n}\mathbf{x}} + \mu_n^{a*}(t)e^{-i\mathbf{n}\mathbf{x}}\right)p_{ij}^a \tag{9}$$

where the constant polarization tensors p_{ii}^a fulfill the conditions $p_{ik}^a n^k = 0$, $p_i^a i = 0$.

Now one should substitute (8) into the field equations (3) and linearize them with respect to $h_{ij}^{(1)}$. It is clear that the left-hand side of eq. (3) is simply the usual D'Alembert differential operator applied to $h_{ij}^{(1)}$. At the same time, the right-hand side of eq. (3) will contain the products of $h_{\mu\nu}^{(0)}$ and $h_{\mu\nu}^{(1)}$ since all the nonlinearities are collected there.

For a given perturbation with the wave vector **n** and for each of the two polarization components, the field equations reduce to a single equation for the time-dependent function $\mu(t)$ (indices n and a are omitted):

 $\ddot{\mu} + n^2 \mu = \left[\frac{n^2 (a^2 - 1)}{a^2} + \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] \mu - \frac{\dot{a}}{a} \dot{\mu}$ (10)

where $n^2 = n^{1^2} + n^{2^2} + n^{3^2}$. If there is no "pump" field (5), that is a(t) = 1, the right-hand side of eq. (10) vanishes. It is worth noting that in the gravitational case, in contrast to electrodynamics, there is not any dimensional coupling constant between the "pump" and "wave" fields, the strength of the coupling is regulated by the rate of variability of the "pump" field.

It is convenient to introduce a new time coordinate η related to t by $d\eta = a(t)^{-1}dt$, and to denote the η -time derivative by prime. Equation (10) gets an especially simple form (Ref. 5):

$$\mu'' + \left(n^2 - \frac{a''}{a}\right)\mu = 0\tag{11}$$

which makes it possible to treat the problem as a problem for a parametrically excited oscillator. The notion of squeezing appears quite naturally.

SQUEEZED VACUUM STATES OF RELIC GRAVITONS

In some cosmologically interesting and realistic situations, the function a''/a goes asymptotically to zero for $\eta \to -\infty$ and $\eta \to +\infty$. In the asymptotic regions $\eta \to -\infty$ and $\eta \to +\infty$, solutions to eq. (11) are very simple: $\mu(\eta) \sim e^{\pm in\eta}$ The general complex solution to eq. (11) can be presented in the form

$$\mu(\eta) = a\xi(\eta) + b^{+}\xi^{*}(\eta) \tag{12}$$

where $\xi(\eta)$ and $\xi^*(\eta)$ are complex-conjugated normalized base functions. The same general solution can be decomposed over other base functions $\chi(\eta)$ and $\chi^*(\eta)$:

$$\mu(\eta) = c\chi(\eta) + d^{+}\chi^{*}(\eta) . \tag{13}$$

One can choose the base functions in such a way that

$$\xi(\eta) \to \frac{1}{\sqrt{2n}} e^{-in\eta} \quad \text{for} \quad \eta \to -\infty$$

and

$$\chi(\eta) \to \frac{1}{\sqrt{2n}} e^{-in\eta} \quad \text{for} \quad \eta \to +\infty \ .$$

Since (12) and (13) describe the same solution, their coefficients are related:

$$a = uc + vd^{+}, \quad b^{+} = v^{*}c + u^{*}d^{+},$$
 (14)

where $|u|^2 - |v|^2 = 1$.

For a quantized field, the coefficients a, b^+, c, d^+ have the meaning of the creation and annihilation operators and the relations (14) are called the Bogoliubov transformations.

Complex numbers u, v can be parameterized by the three real numbers $r, \theta, \varphi, r \geq 0$:

$$u = e^{-i\theta} chr, v = -e^{i(\theta + 2\varphi)} shr.$$
 (15)

The transformations (14) can also be presented in the form:

$$a = R^+ S^+ c S R, \qquad b^+ = R^+ S^+ d^+ S R,$$
 (16)

where $S(r,\varphi)$ and $R(\theta)$ are unitary operators:

$$S(r,\varphi) = \exp[r(e^{-2i\varphi}cd - e^{2i\varphi}c^{+}d^{+})]$$

$$R(\theta) = \exp[-i\theta c^{+}c - i\theta d^{+}d].$$

In the theory of squeezed quantum states, the operator $S(r,\varphi)$ is called the two-mode squeeze operator, the operator $R(\theta)$ is the two-mode rotation operator, r is the squeeze parameter and φ is the squeeze angle (see, for instance, Ref. 6).

As a result of evolution, the two-mode vacuum state $|0,0\rangle$ transforms into a two-mode squeezed vacuum state:

$$|SS\rangle_2 = S(r,\varphi)|0,0\rangle$$
.

The two modes under discussion are two waves with the same frequency but propagating in opposite directions. In the field of quantum optics, one is usually interested in the temporal fluctuations of the light field, so that the spatial distribution of the field is not always important. However, in cosmology, we need to know the complete space-time distribution of the gravity-wave field. For this aim we augment the time-dependent functions $\mu(\eta)$ with the spatial functions $U(\mathbf{x})$. For every n-mode contribution and a given polarization component, one has:

$$h_{\mathbf{n}} = \mu_{n,1} U_{\mathbf{n},1} + \mu_{n,2} U_{\mathbf{n},2}$$
.

One can work, for example, with complex functions $U_{\mathbf{n},1} = K e^{i\mathbf{n}\mathbf{x}}$, $U_{\mathbf{n},2} = K e^{-i\mathbf{n}\mathbf{x}}$ (and complex conjugated μ : $\mu_{n,2} = \mu_{n,1}^*$) or with real functions $U_{\mathbf{n},1} = K \cos \mathbf{n}\mathbf{x}$, $U_{\mathbf{n},2} = K \sin \mathbf{n}\mathbf{x}$ (and real μ), where K is a normalization constant. Classically, this corresponds to the decomposition of the field over traveling or standing waves. For the field operators, one writes

$$h_{\mathbf{n}} = (a\xi + b^{+}\xi^{*}) K e^{i\mathbf{n}\mathbf{x}} + (a^{+}\xi^{*} + b\xi) K e^{-i\mathbf{n}\mathbf{x}},$$

in the first case, and

$$h_{\mathbf{n}} = \left(b_1 \xi + b_1^+ \xi^*\right) \sqrt{2} K \cos \mathbf{n} \mathbf{x} + \left(b_2 \xi + b_2^+ \xi^*\right) \sqrt{2} K \sin \mathbf{n} \mathbf{x} , \qquad (17)$$

in the second case. Transition between the two descriptions is fulfilled by the transformation

$$\frac{a+b}{\sqrt{2}} = b_1 \ , \quad \frac{i(a-b)}{\sqrt{2}} = b_2 \ . \tag{18}$$

In terms of the theory of squeezed states, it means that, in the second case, one will be dealing with a pair of one-mode squeezed states instead of a single two-mode state. Indeed, under the transformation

(18) the two-mode squeeze operator factorizes into a product of the two one-mode squeeze operators $S_1(r,\varphi)$, so that instead of (16) one will have

$$b_1 = R_1^+ S_1^+ c_1 S_1 R_1$$

where $S_1(r,\varphi)$ is the one-mode squeeze operator

$$S_1(r,\varphi) = \exp\left[\frac{r}{2}(e^{-2i\varphi}c_1^2 - e^{2i\varphi}c_1^{+2})\right]$$

and $R_1(\theta)$ is the one-mode rotation operator

$$R_1(\theta) = \exp(-i\theta c_1^+ c_1) .$$

The operator b_2 transforms in precisely the same manner and is associated with the other squeezed mode.

As a result of evolution, the one-mode vacuum state transforms into a one-mode squeezed state.

In what follows, we will base our analysis on the representation (17) and on the one-mode squeezed states. The annihilation and creation operators will be denoted by b and b^+ .

The classical equations of motion (11) can be derived from the Hamiltonian

$$H = \frac{1}{2} \left[p^2 + \frac{a'}{a} (\mu p + p \mu) + n^2 \mu^2 \right]$$

where p is the momentum, canonically conjugated to the coordinate μ , $p = \mu' - \frac{a'}{a}\mu$. In a standard manner, by introducing the annihilation and creation operators b, b^+ :

$$b = \sqrt{\frac{n}{2}} \left(\mu + i \frac{p}{n} \right) , b^+ = \sqrt{\frac{n}{2}} \left(\mu - i \frac{p}{n} \right) ,$$

one can present the Hamiltonian in the form

$$H = nb^{+}b + \sigma(\eta)b^{+^{2}} + \sigma^{*}(\eta)b^{2}$$
 (19)

where the coupling function $\sigma(\eta)$ is $\sigma(\eta) = ia'/2a$, and the Planck constant $\hbar = 1$. Note that the Hamiltonian (19) belongs to the class of Hamiltonians that characterize a number of physical processes (Ref. 7). However, in most of them the function $\sigma(t)$ has a specific form $\sigma(t) = \sigma e^{-2i\omega t}$ where σ is a constant, albeit in our case $\sigma(\eta)$ is a more general function of time.

The Heisenberg equations of motion, following from this Hamiltonian, have the form

$$i\frac{db}{d\eta} = nb + i\frac{a'}{a}b^+$$
, $-i\frac{db^+}{d\eta} = nb^+ - i\frac{a'}{a}b$.

Their solution is

$$b(\eta) = u(\eta)b_0 + v(\eta)b_0^+$$
, $b^+(\eta) = u^*(\eta)b_0^+ + v^*(\eta)b_0$,

where b_0 , b_0^+ are the initial values of $b(\eta)$, $b^+(\eta)$ (Schrödinger operators) and the complex functions $u(\eta)$ and $v(\eta)$ satisfy the equations

$$iu' = nu + i\frac{a'}{a}v^*, -iv^{*'} = nv^* - i\frac{a'}{a}u, \quad u(0) = 1, \quad v(0) = 0.$$
 (20)

It follows from these equations that $u + v^*$ satisfies the equation identical to eq. (11):

$$(u+v^*)''+(n^2-\frac{a''}{a})(u+v^*)=0.$$

As for the function $u - v^*$, it can be found from the relation

$$-in(u-v^*) = (u+v^*)' - \frac{a'}{a}(u+v^*).$$

By substituting eq. (15) into eq. (20) one can find equations for the time-dependent parameters $r(\eta)$, $\theta(\eta)$, $\varphi(\eta)$:

$$r' = -\frac{a'}{a}\cos 2\varphi$$

$$\theta' = n - \frac{a'}{a}\sin 2\varphi \, thr$$

$$\varphi' = -n - \frac{a'}{a}\sin 2\varphi \, cth2r \,. \tag{21}$$

Solutions to these equations determine the precise form of evolution of the initial vacuum state into a one-mode squeezed vacuum state. Statistical properties of the final state depend on the numerical values of r and φ in a well-known way (see, for instance, Refs. 6,7).

A possible way of calculating $r(\eta)$ and $\varphi(\eta)$ (Ref. 8) for a given gravitational field $a(\eta)$ relies on the observation that the complex function $B(\eta)$, where

$$\left(\frac{u-v^*}{u+v^*}\right)^* \equiv \frac{chr + e^{2i\varphi}shr}{chr - e^{2i\varphi}shr} = \frac{2}{n}B(\eta) , \qquad (22)$$

satisfies the equation

$$B' = i\frac{n^2}{2} - 2\frac{a'}{a}B - 2iB^2$$

with the solution

$$B(\eta) = -\frac{i}{2} \frac{(\mu/a)'}{\mu/a}$$

where $\mu(\eta)$ obeys eq. (11). The properly chosen solutions to eq. (11) define $B(\eta)$ and allow one to find $r(\eta)$ and $\varphi(\eta)$ from eq. (22). The meaning of the function $B(\eta)$ is that it determines the Gaussian wave function $\Psi(\mu,\eta) \sim \exp(-B(\eta)\mu^2)$ which is a solution to the Schrödinger equation in the coordinate representation.

The parameters $r(\eta)$, $\varphi(\eta)$ can be calculated (Ref. 8) for a cosmological model which includes three sequential stages of expansion: inflationary $(a(t) \sim e^{H_o t})$, radiation-dominated $(a(t) \sim t^{1/2})$ and matter-dominated $(a(t) \sim t^{2/3})$. It can be shown that the present-day values of the squeeze parameter r range from $r \approx 1$, for frequencies $\nu \approx 10^8$ Hz, up to $r \approx 120$, for frequencies $\nu \approx 10^{-18}$ Hz. In the

frequency interval $\nu \approx 10^{-1}-10^{-3}$ Hz, accessible for the planned Laser Interferometer Gravitational-Wave Observatory in Space (Ref. 9), the squeeze parameter r reaches large values of order 40-50.

As for the parameter $\varphi(\eta)$, it can be shown to have the form $\varphi \approx -n\eta + \varphi_0$, where φ_0 is a constant. This behaviour can already be envisaged from eq. (21) for φ' , since, in the asymptotic region $\eta \to +\infty$, one has $\left|\frac{a'}{a}\right| \ll n$ and $cth2r \approx 1$.

RELIC GRAVITONS: A STOCHASTIC COLLECTION OF STANDING WAVES

As we see, the cos nx and sin nx modes in the representation (17), evolve into a strongly squeezed vacuum state. The mean number of quanta $\langle N \rangle$ and its variance $\langle (\Delta N)^2 \rangle$ are determined by the squeeze parameter r:

$$\langle N \rangle = sh^2r, \quad \langle (\Delta N)^2 \rangle = \frac{1}{2}sh^22r.$$

The mean values of $\hat{\mu}$ and \hat{p} are equal to zero, but their variances do not vanish:

$$\langle (\Delta \hat{\mu})^2 \rangle = \frac{1}{2n} (ch2r - sh2r\cos 2\varphi), \quad \langle (\Delta \hat{p})^2 \rangle = \frac{n}{2} (ch2r + sh2r\cos 2\varphi).$$

In order to relate the rigorous quantum-mechanical treatment, described above, with the notions of random classical waves, one can use the Wigner function formalism. It allows one to derive the distributions of the random variables A and ϕ entering the classical expression for μ :

$$\mu = A\sin(-n\eta + \phi). \tag{23}$$

It can be shown (see Ref. 8) that, in the limit of large r, the Gaussian distribution for ϕ is very narrow, like a δ -function. It is concentrated near the values

$$\phi = \varphi_0 + \pi \ell \tag{24}$$

where φ_0 is a constant, the same for all unit vectors \mathbf{n}/n , and $\ell=0,\pm 1,\ldots$

A similar conclusion can be reached in a simpler, though perhaps less rigorous way. Let us consider the ratio $\langle (\Delta \hat{\mu})^2 \rangle / \langle (\Delta \hat{p})^2 \rangle$. For large r, this number is approximated by

$$\langle (\Delta \hat{\mu})^2 \rangle / \langle (\Delta \hat{p})^2 \rangle \approx \frac{1}{n^2} t g^2 \varphi$$
.

For a classical expression (23), this ratio corresponds to the number n^{-2} $tg^2(-n\eta + \phi)$. From their comparison, and taking into account the fact that $\varphi \approx -n\eta + \varphi_0$, one can derive eq. (24). The very small variance of the phase, $\Delta \phi$, is, of course, consistent with the large variance of the number of quanta, $\langle (\Delta N)^2 \rangle$.

The negligibly small variance of the phase distribution leads to an important result: every pair of $\cos nx$, $\sin nx$ modes forms together a standing wave. Indeed, let us consider a given n. The corresponding terms, contributing to the total wave-field $h(\eta, x, y, z)$, can be written in the general form:

$$h_{\mathbf{n}} = A_1 \sin(-n\eta + \phi_1) \cos \mathbf{n} \mathbf{x} + A_2 \sin(-n\eta + \phi_2) \sin \mathbf{n} \mathbf{x}. \tag{25}$$

The amplitudes A_1 and A_2 are taken from a broad Gaussian distribution and are, in general, different. However, the phases ϕ_1 and ϕ_2 are taken from a very narrow Gaussian distribution and are essentially fixed and equal up to $\pm \pi$. Because of that, expression (25) can be written as a product of a function of time and a function of spatial coordinates:

$$h_{\mathbf{n}} = \pm \sin(-n\eta + \varphi_0)(A_1 \cos \mathbf{n} \mathbf{x} \pm A_2 \sin \mathbf{n} \mathbf{x}). \tag{26}$$

In other words, expression (26) describes a standing wave. A characteristic feature of a standing wave pattern is that the field vanishes all over the space at every half of the period. The randomness of the wave-field is displayed in its spatial functions $A_1 \cos nx \pm A_2 \sin nx$. This is why we say that the relic gravitational waves are present now in the cosmos in the form of a stochastic collection of standing waves.

The total field $h(\eta, \mathbf{x})$ is obtained by summing over all n-mode contributions (26). Of course, the total field loses the property of vanishing at some moments of time, because the various $\sin(-n\eta + \varphi_0)$ factors have different arguments. However, the difference in the arguments is not random, but deterministic. For instance, if at some moment of time $\eta = \eta_0$ the component $h_n(\eta, \mathbf{x})$ vanishes, the same will be true for all other components $h_m(\eta, \mathbf{x})$, where $m = n(1 + k/\ell)$ and k/ℓ is an arbitrary rational number. Hopefully, this property can somehow be used in a specific strategy of observational discrimination of relic gravitational waves from stochastic gravitational waves of a different origin. I think that the inevitable "squeezing" of relic gravitational waves (and other primordial fluctuations of quantum-mechanical origin) can manifest itself in a variety of circumstances, not all of which are foreseeable at the moment.

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